Approximations, asymptotics and inference for continuous-time locally stationary processes

Bennet Ströh

Imperial College London

Joint work with Annemarie Bitter and Robert Stelzer

Summary

Locally stationary process

$$Y_N(t) = \int_{-\infty}^{Nt} e^{-\int_s^{Nt} a(\frac{\tau}{N})d\tau} L(ds).$$

Stationary approximation

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***M*-estimation**

$$\hat{\vartheta}_N = \arg\min_{\vartheta \in \Theta} \frac{\delta_N}{b_N} \sum_{i=-m_N}^{m_N} K\left(\frac{\tau_i^N - u}{b_N}\right) \Phi\left(\left(Y_N(\tau_{i+1-k}^N)\right)_{k \in \mathbb{N}_0}, \vartheta\right)$$

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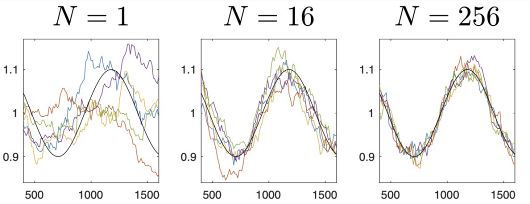

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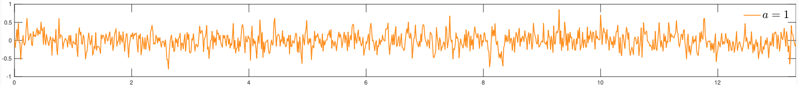

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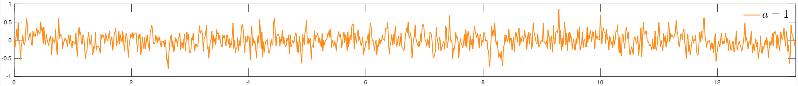

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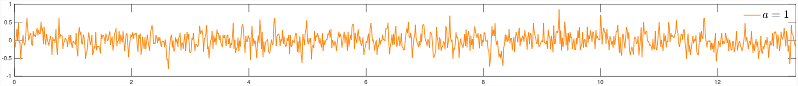

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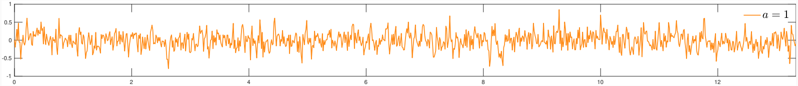

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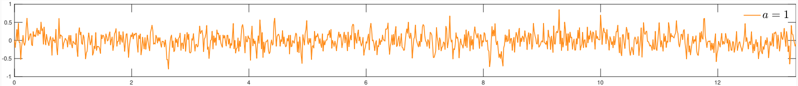

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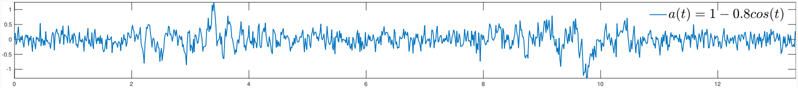

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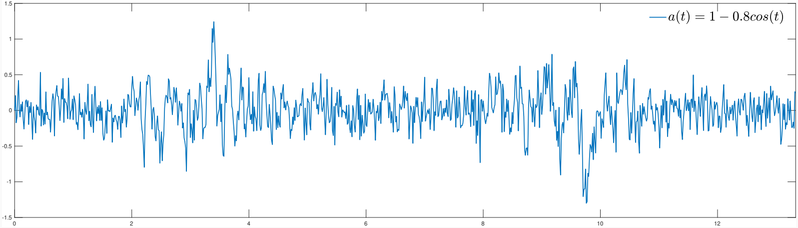

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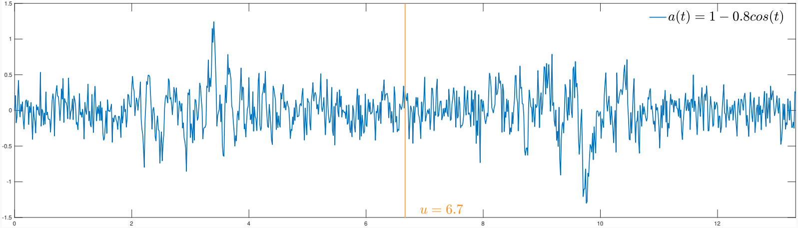

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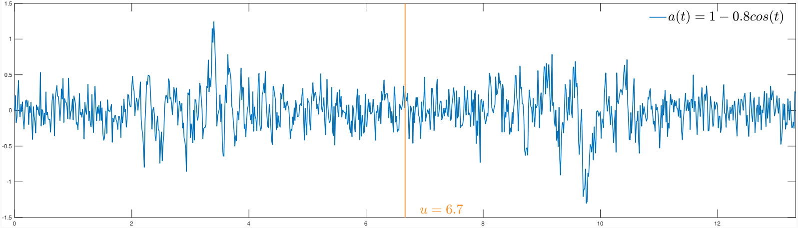

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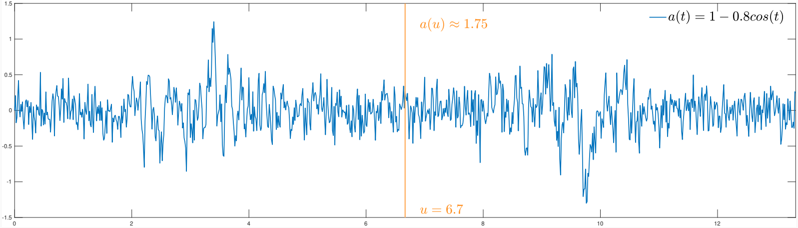

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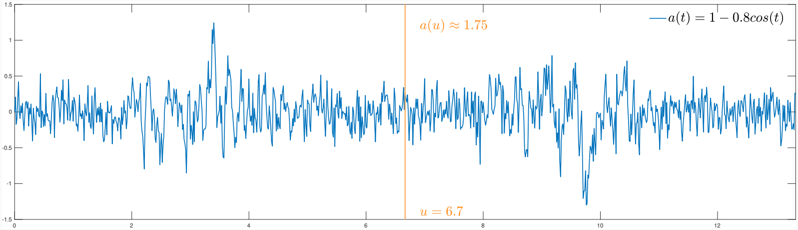

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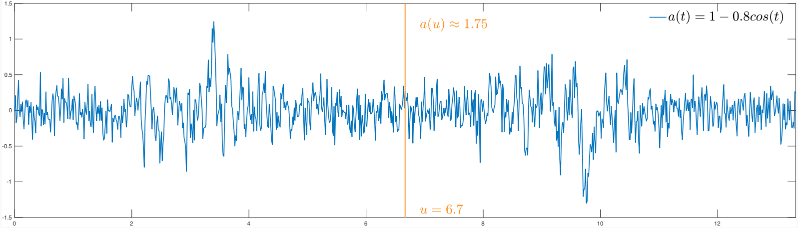

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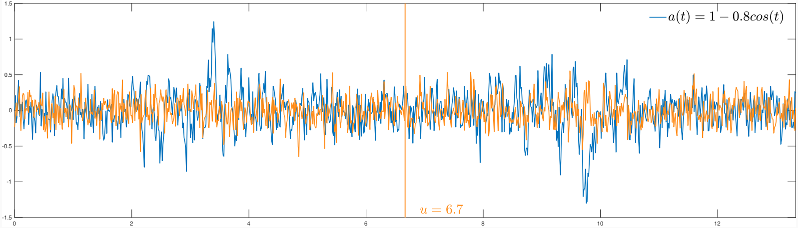

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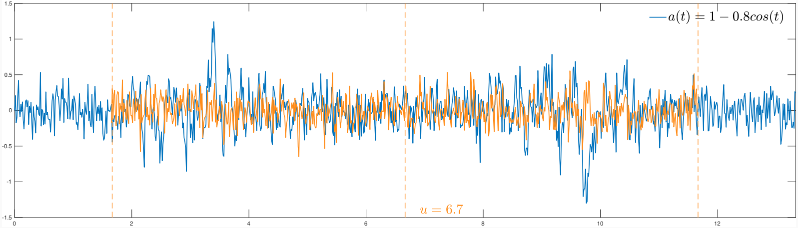

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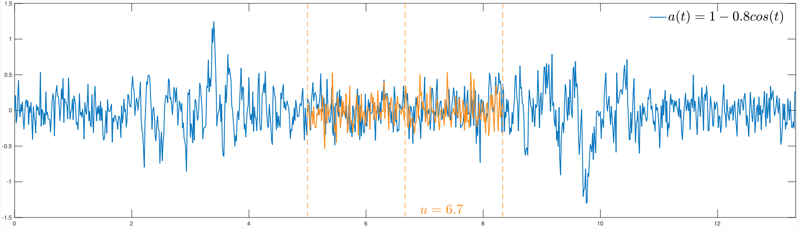

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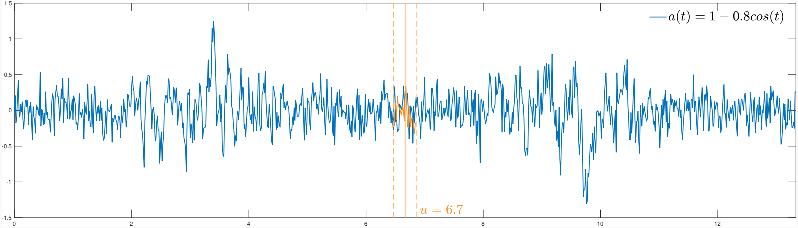

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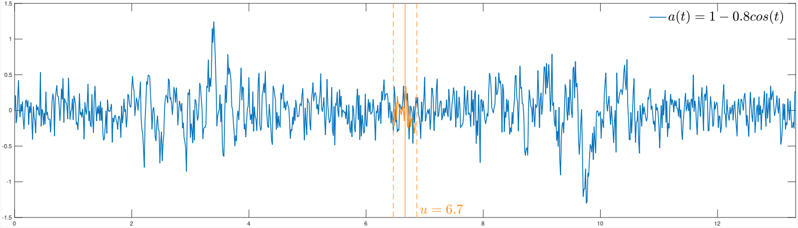

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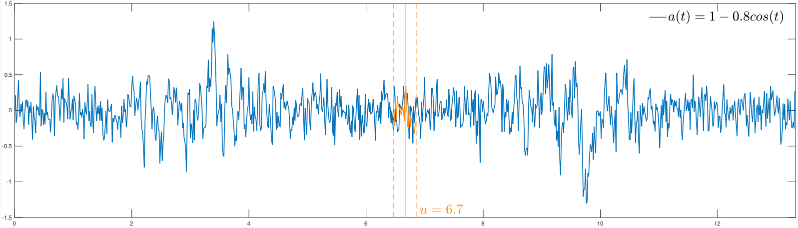

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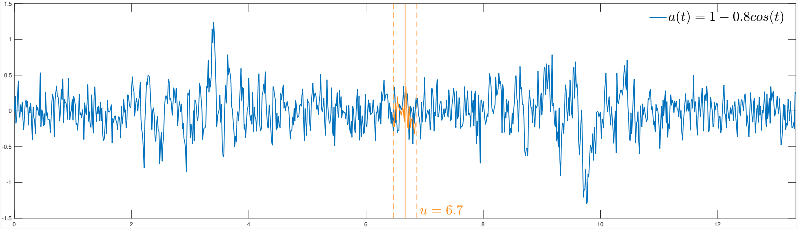

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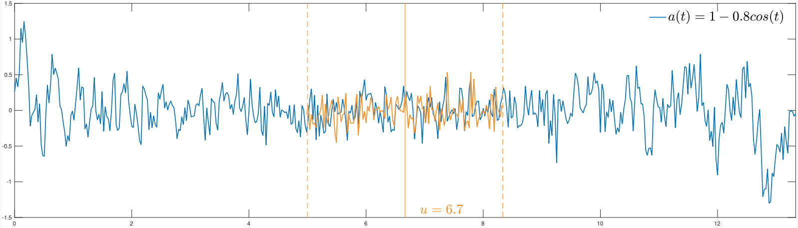

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- K is localizing kernel, i.e. bounded, $\int_{\mathbb{R}} K(x)dx = 1$, bounded variation and bounded support

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where $\sigma(u)^2 = \frac{1}{2}E[\tilde{Y}_u(0)^2] + \sum_{k=1}^{\infty} E[\tilde{Y}_u(0)\tilde{Y}_u(k\delta)] > 0$.

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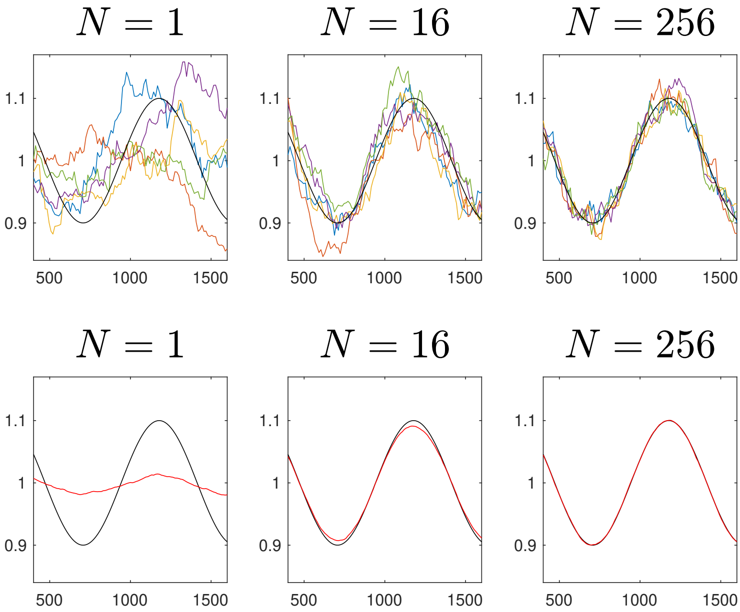
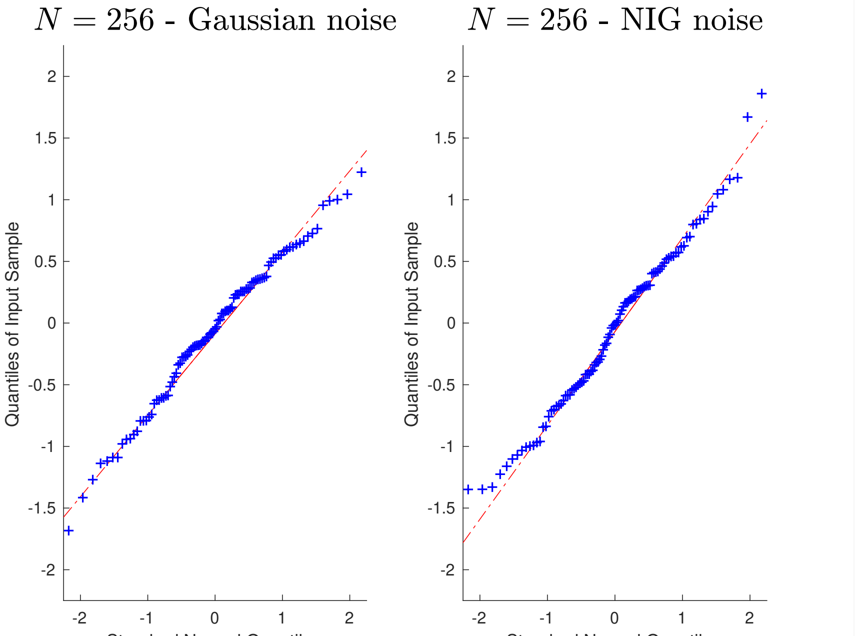
- Then $\hat{\vartheta}_N \xrightarrow[N\to\infty]{P} \vartheta^*$and $\sqrt{\frac{b_N}{\delta_N}}\left(\hat{\vartheta}_N - \vartheta^*\right) \xrightarrow[N\to\infty]{d} \mathcal{N}(0,\Sigma)$, where Σ is some known covariance matrix.

Simulation Study: Least Squares for tvCAR(1)



Simulation Study: Least Squares for tvCAR(1)



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